# Algebra, Linear Algebra, and Numerical Analysis

by

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# 1 Logarithms

$$\log_b x = c \iff x = b^c$$

The *common logarithm* is

$$\log(x) = \log_{10}(x)$$

The *Naperian* or *natural logarithm* is

$$\ln(x) = \log_e(x), \qquad e = 2.7183\ldots$$

Also,

$$\operatorname{anti}\log(\log(x)) = x$$

## 1.1 Properties

$$\begin{aligned}
\log_b(x) &= \frac{\log_a(x)}{\log_a(b)} \\
\log_b(b^n) &= n \\
\log_b(b) &= 1 \\
\log(1) &= 0 \\
\log(x^c) &= c\log(x) \\
\log(xy) &= \log(x) + \log(y) \\
\log\left(\frac{x}{y}\right) &= \log(x) - \log(y)
\end{aligned}$$

## 1.2 Examples

**Example 1** *Find* $\log_{10}(0.00001)$.

Solution: We have

$$\begin{aligned} \log_{10}(0.00001) &= c \\ &\Longleftrightarrow 10^c = 0.00001 = 10^{-5} \\ &\Rightarrow c = -5 \end{aligned}$$

**Example 2** *Find the common logarithm of* $1000^4$.

Solution: We have

$$\begin{aligned} \log_{10}\left(1000^4\right) &= \log_{10}\left(\left(10^3\right)^4\right) \\ &= \log_{10}\left(10^{12}\right) \\ &= 12\ \log_{10}(10) \\ &= 12 \end{aligned}$$

**Example 3** *Find the Naperian logarithm of* $e^{1+x-y}$.

Solution: We have

$$\begin{aligned} \ln\left(e^{1+x-y}\right) &= (1+x-y)\ln(e) \\ &= 1+x-y \end{aligned}$$

# 2 Complex Numbers

$$a + ib, \quad i^2 = -1$$

## 2.1 Multiplication/Division/Conjugation

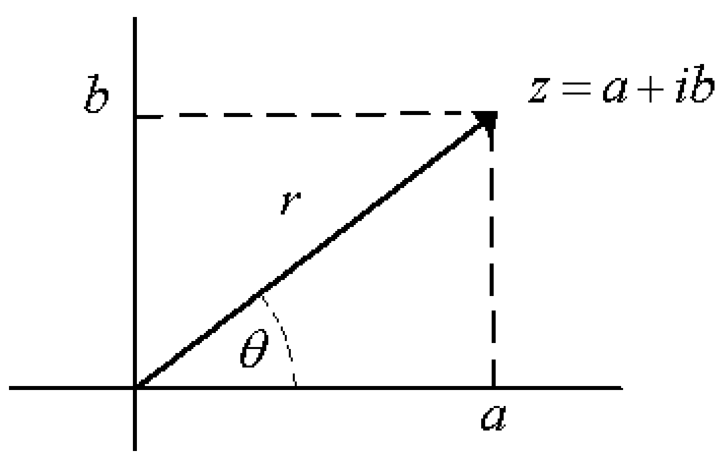
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}$$

$$|z| = a^2 + b^2, \qquad \overline{z} = a - ib$$

## 2.2 Polar Form

$$z = r(\cos\theta + i\sin\theta)$$



## Conversions

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

## Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

## Multiplication/Division/Powers/Roots in Polar

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

$$z^n = r^n\left[\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right]$$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + k\frac{360^0}{n}\right) + i\sin\left(\frac{\theta}{n} + k\frac{360^0}{n}\right)\right]$$

$$k = 0, \ldots, n-1$$

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

## 2.3 Examples

**Example 4** *Find the polar form of* $z = 6 + 7i$.

Solution: We have

$$\begin{aligned} r &= \sqrt{6^2 + 7^2} = 9.2195 \\ \theta &= \arctan(7/6) = 0.86217 \,\text{rad} = 49.40^\circ \\ &\Rightarrow z = 9.2195\,(\cos(49.40^\circ) + i \sin(49.40^\circ)) \end{aligned}$$

**Example 5** *Find* $\sqrt[3]{1+i}$

Solution: We have

$$\begin{aligned} \sqrt[3]{1+i} &= \sqrt[3]{2}\left[\cos\left(\frac{\pi/4 + 2k\pi}{3}\right) + i \sin\left(\frac{\pi/4 + 2k\pi}{3}\right)\right] \\ k &= 0, \quad \sqrt[3]{2}\left[\cos\left(\frac{\pi/4}{3}\right) + i \sin\left(\frac{\pi/4}{3}\right)\right] \\ &= 1.084215081 + 0.2905145554i \\ k &= 1, \quad \sqrt[3]{2}\left[\cos\left(\frac{\pi/4 + 2\pi}{3}\right) + i \sin\left(\frac{\pi/4 + 2\pi}{3}\right)\right] \\ &= -0.7937005260 + 0.7937005260i \\ k &= 1, \quad \sqrt[3]{1^2 + 1^2}\left[\cos\left(\frac{\pi/4 + 4\pi}{3}\right) + i \sin\left(\frac{\pi/4 + 4\pi}{3}\right)\right] \\ &= -0.2905145554 - 1.084215081i \end{aligned}$$

# 3 Matrices

## 3.1 Operations

**Addition:** $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

**Example 6**

$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & -4 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

**Scalar Multiplication:** $cA = c\,[a_{ij}] = [ca_{ij}]$

**Example 7**

$$2\begin{bmatrix} 1 & 0 \\ -3 & 4 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -6 & 8 \\ 10 & -2 \end{bmatrix}$$

**Matrix Multiplication:** If $A_{m\times k}$, $B_{k\times n}$, then $(AB)_{m\times n}$

$$A = [a_{mn}]\,, \quad B = [b_{rq}]\,, \quad AB = [c_{ij}]$$
$$c_{ij} = a_{i1}\,b_{1j} \;+\; a_{i2}\,b_{2j} \;+\cdots+ a_{ik}\,b_{kj}$$

**Example 8**

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 \\ -2 & 4 & 5 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 6 \\ 4 & 14 & 9 \end{bmatrix}$$

**Example 9**

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -1 \\ -15 & -3 & 3 \\ 20 & 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = [-2] \quad \text{So } AB \neq BA$$

## 3.2 Identity

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \ldots, \quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$A_{m\times n} I_n = A_{m\times n} = I_m A_{m\times n}$$

## 3.3 Transpose

The transpose of $A = [a_{ij}]$, is $A^T = [a_{ji}]$.

**Example 10**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

## 3.4 Introduction to Determinants

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

**Example 11**

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

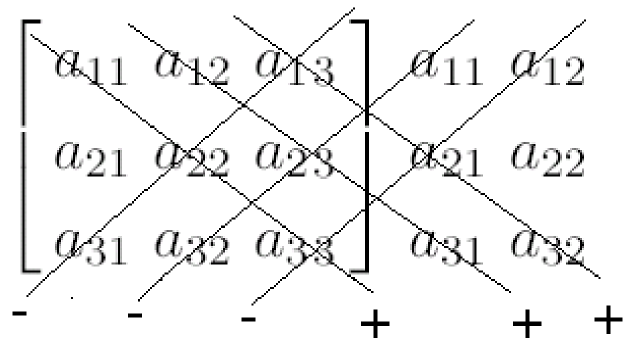
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

**Example 12**

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & -1 \end{vmatrix} &= 1 \begin{vmatrix} 0 & -2 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ &= = 1 \cdot 4 - 2 \cdot (-1) + 0 \cdot 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & -2 & -1 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} \\ &= 6 \end{aligned}$$

**The Sarrus Scheme**



$$a_{11}\,a_{22}a_{33} + a_{12}\,a_{23}a_{31} + a_{13}\,a_{21}a_{32}$$
$$-a_{13}a_{22}a_{31} - a_{11}\,a_{23}a_{32} - a_{12}a_{21}a_{33}$$

**Warning: This works only for** $3 \times 3$ **determinants!**

## 3.5 Cofactors

The $(i, j)$ **minor**

$$M_{ij}$$

of a square matrix $A$ is the determinant obtained by deleting the $i$th row and the $j$th column.

The signed minor $(-1)^{i+j}M_{ij}$ is called the $(i, j)$ **cofactor**, of $A$ and is denoted by $C_{ij}$.

$$C_{ij} = (-1)^{i+j}M_{ij}$$

The signs $(-1)^{i+j}$ follow a checkerboard pattern of $\pm$'s

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The **matrix of cofactors** is $[C_{ij}]$

**Example 13** *The cofactors of* $A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix}$ *are*

$$M_{11} = \begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix} = 9 \qquad C_{11} = (-1)^{1+1} M_{11} = 9$$

$$M_{12} = \begin{vmatrix} 4 & -2 \\ -5 & 3 \end{vmatrix} = 2 \qquad C_{12} = (-1)^{1+2} M_{12} = -1 \cdot 2 = -2$$

$$M_{13} = \begin{vmatrix} 4 & 3 \\ -5 & 0 \end{vmatrix} = 15 \qquad C_{13} = (-1)^{1+3} M_{13} = 15$$

$$M_{21} = \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6 \qquad C_{21} = (-1)^{2+1} M_{21} = -1 \cdot 6 = -6$$

$$M_{22} = \begin{vmatrix} -1 & 2 \\ -5 & 3 \end{vmatrix} = 7 \qquad C_{22} = (-1)^{2+2} M_{22} = 7$$

$$M_{23} = \begin{vmatrix} -1 & 2 \\ -5 & 0 \end{vmatrix} = 10 \qquad C_{23} = (-1)^{2+3} M_{23} = -1 \cdot 10 = -10$$

$$M_{31} = \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} = -10 \qquad C_{31} = (-1)^{3+1} M_{31} = -10$$

$$M_{32} = \begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = -6 \qquad C_{32} = (-1)^{3+2} M_{32} = (-1)(-6) = 6$$

$$M_{33} = \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} = -11 \qquad C_{33} = (-1)^{3+3} M_{33} = -11$$

**Example 14** *The matrix of cofactors of $A$ in the previous example is*

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 9 & -2 & 15 \\ -6 & 7 & -10 \\ -10 & 6 & -11 \end{bmatrix}$$

## 3.6 Cofactor Expansion of Determinant

(1) **Cofactor Expansion about the ith row** The determinant of $A$ can be expanded about the $i$th row in terms of the cofactors as follows.

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(2) **Cofactor Expansion about the jth column** The determinant of $A$ can be expanded about the $j$th column in terms of the cofactors as follows.

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

**Example 15**

$$\begin{aligned}\det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\\ &= (-1)9 + 2(-2) + 2\cdot 15 = 17\end{aligned}$$

$$\begin{aligned}\det A &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}\\ &= 4(-6) + 3\cdot 7 + (-2)(-10) = 17\end{aligned}$$

$$\begin{aligned}\det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\\ &= (-5)(-10) + 0\cdot 6 + 3(-11) = 17\end{aligned}$$

$$\begin{aligned}\det A &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}\\ &= (-1)9 + 4(-6) + (-5)(-10) = 17\end{aligned}$$

$$\begin{aligned}\det A &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}\\ &= 2(-2) + 3\cdot 7 + 0\cdot 6 = 17\end{aligned}$$

$$\begin{aligned}\det A &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}\\ &= 2\cdot 15 + (-2)(-10) + 3(-11) = 17\end{aligned}$$

## 3.7 Properties of Determinants

(1) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(2) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ kb_1 & kb_2 & kb_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(3) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(4) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 + b_1 & ka_2 + b_2 & ka_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(5) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

(6) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

(7) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

(8) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 + lc_1 & ka_2 + lc_2 & ka_3 + lc_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

(9) Cauchy’s Theorem

$$\det(AB) = \det(A)\det(B)$$

(10) For $A_{n\times n}$

$$\det(kA) = k^n \det(A)$$

## 3.8 The Adjoint

The transpose $[C_{ji}]$ of the cofactor matrix $[C_{ij}]$ of a square matrix $A$ is the **adjoint of** $A$ and it is denoted by $\mathrm{Adj}(A)$.

$$\mathrm{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

**Example 16** *Find the adjoint of $A$, where*

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix}$$

Solution: In Example 14 we found the cofactors of $A$ to be

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 9 & -2 & 15 \\ -6 & 7 & -10 \\ -10 & 6 & -11 \end{bmatrix}$$

Hence,

$$\mathrm{Adj}(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 9 & -6 & -10 \\ -2 & 7 & 6 \\ 15 & -10 & -11 \end{bmatrix}$$

Basic Property

$$A\,\mathrm{Adj}(A) = \det(A) I_n = \mathrm{Adj}(A)\,A$$

### 3.9 The Inverse

An $n \times n$ matrix $A$ is **invertible**, if there exists a matrix $A^{-1}$ such that

$$AA^{-1} = I \qquad \text{and} \qquad A^{-1}A = I$$

In such case $A^{-1}$ is called an **inverse** of $A$. If is a fact that if $A^{-1}$ exists, it is unique.

If no inverse $A^{-1}$ exists for $A$, then we say that $A$ is **noninvertible**. Another name for invertible is **nonsingular** and another name for noninvertible is **singular**.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det(A) = ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

BASIC FACTS

(1) $A$ is invertible if and only if $\det(A) \neq 0$

(2) Let $A$ be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A) \tag{1}$$

**Example 17** *Find $A^{-1}$, where $A$ is as in Example 16.*

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix}$$

Solution: We have

$$\det(A) = \begin{vmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{vmatrix} = 17$$

Hence, by Example 14, and (1)

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)}\mathrm{Adj}(A) \\ &= \frac{1}{17}\begin{bmatrix} 9 & -6 & -10 \\ -2 & 7 & 6 \\ 15 & -10 & -11 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{17} & -\frac{6}{17} & -\frac{10}{17} \\ -\frac{2}{17} & \frac{7}{17} & \frac{6}{17} \\ \frac{15}{17} & -\frac{10}{17} & -\frac{11}{17} \end{bmatrix} \\ &= \begin{bmatrix} 0.52941 & -0.35294 & -0.58824 \\ -0.11765 & 0.41176 & 0.35294 \\ 0.88235 & -0.58824 & -0.64706 \end{bmatrix} \end{aligned}$$

Verification with small numerical error.

$$AA^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0.52941 & -0.35294 & -0.58824 \\ -0.11765 & 0.41176 & 0.35294 \\ 0.88235 & -0.58824 & -0.64706 \end{bmatrix}$$

$$= \begin{bmatrix} 0.99999 & -0.00002 & 0 \\ -0.00001 & 1.0 & -0.00002 \\ 0 & -0.00002 & 1.0 \end{bmatrix}$$

# 4 Linear Systems

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 5 \\ x_1 - 2x_2 &= 6 \\ -3x_1 + x_2 &= 1 \end{aligned} \qquad \begin{aligned} y_1 + y_2 + y_3 &= -2 \\ y_1 - 2y_2 + 7y_3 &= 6 \end{aligned}$$

## 4.1 Square Linear Systems

$$\begin{aligned} -x + 2y + 2z &= -20 \\ 4x + 3y - 2z &= -7 \\ -5x + 3z &= -24 \end{aligned}$$

or in matrix form $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ -7 \\ -24 \end{bmatrix}$$

One way to solve is by using the inverse if it exists

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ &\Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \\ &\Rightarrow I\mathbf{x} = A^{-1}\mathbf{b} \\ &\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \end{aligned}$$

**Example 18** *Solve the system*

$$\begin{aligned} -x + 2y + 2z &= -20 \\ 4x + 3y - 2z &= -7 \\ -5x + 3z &= -24 \end{aligned}$$

Solution: From Example 17, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{9}{17} & -\frac{6}{17} & -\frac{10}{17} \\ -\frac{2}{17} & \frac{7}{17} & \frac{6}{17} \\ \frac{15}{17} & -\frac{10}{17} & -\frac{11}{17} \end{bmatrix} \begin{bmatrix} -20 \\ -7 \\ -24 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ 2 \end{bmatrix}$$

So

$$x = 6, \quad y = -9, \quad z = 2$$

Verification

$$\begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ -9 \\ 2 \end{bmatrix} = \begin{bmatrix} -20 \\ -7 \\ -24 \end{bmatrix}$$

### 4.1 Cramer's Rule

Let $A\mathbf{x} = \mathbf{b}$ be a square system, with

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Let $A_i$ denote the matrix obtained from $A$ by replacing the $i$th column with $\mathbf{b}$.

$$A_i = \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}$$

Cramer's Rule gives *an explicit formula* for the solution of a consistent square system.

**Theorem 1 (Cramer's Rule)** *If* $\det(A) \neq 0$, *then the system*

$$A\mathbf{x} = \mathbf{b}$$

*has a unique solution* $\mathbf{x} = (x_1, \dots, x_n)$ *given by*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \; . \quad .. \, , \; x_n = \frac{\det(A_n)}{\det(A)}$$

**Example 19** *Use Cramer's Rule to solve the system.*

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 + x_3 &= 4 \end{aligned}$$

Solution: We compute the determinant of the coefficient matrix $A$ and the determinants of

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

to get $\det(A) = -4$, $\det(A_1) = -10$, $\det(A_2) = -12$, $\det(A_3) = -14$. Hence,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{5}{2}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = 3, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{7}{2}$$

**Example 20** *Use Cramer's Rule to find the solution to the general linear system, if* $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Solution: Since $|A| = a_{11}a_{22} - a_{12}a_{21} \neq 0$, we have

$$x_1 = \frac{|A_1|}{|A|} = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{|A_2|}{|A|} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

# 5 Vectors

Vector addition $\mathbf{a} + \mathbf{b}$

**Example 21**

$$\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ -7 \\ -13 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -9 \end{bmatrix}$$

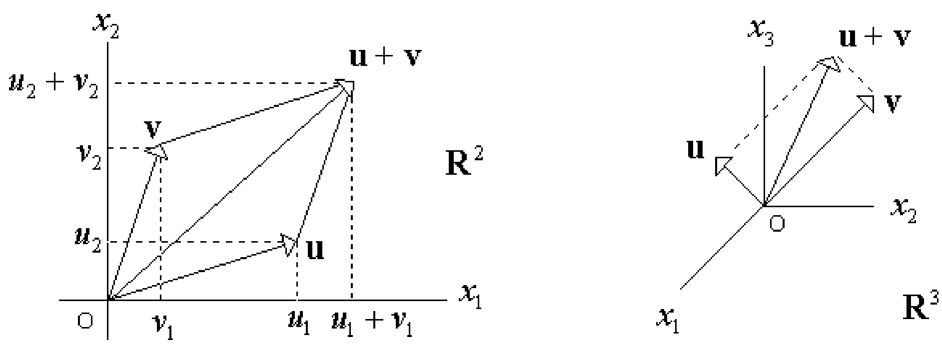



**Fig. :** The parallelogram law for vector addition.

Scalar multiplication $c\mathbf{a}$

**Example 22**

$$5 \begin{bmatrix} 8 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 40 \\ -15 \\ 35 \end{bmatrix}$$

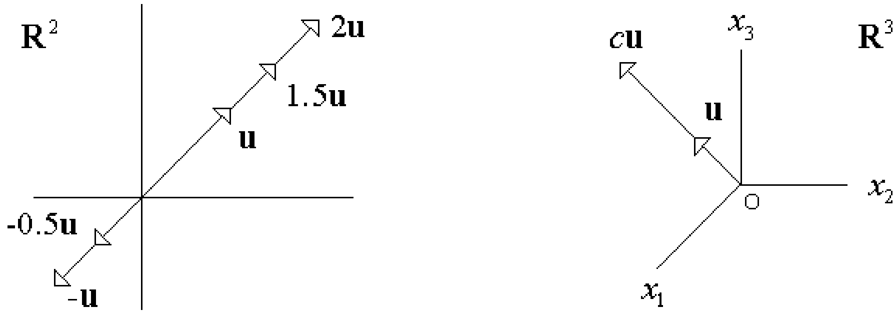



**Fig. :** Scalar products.

The standard basis vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ are denoted by $\mathbf{i}, \mathbf{j}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\mathbf{i} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{j} = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{k} = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

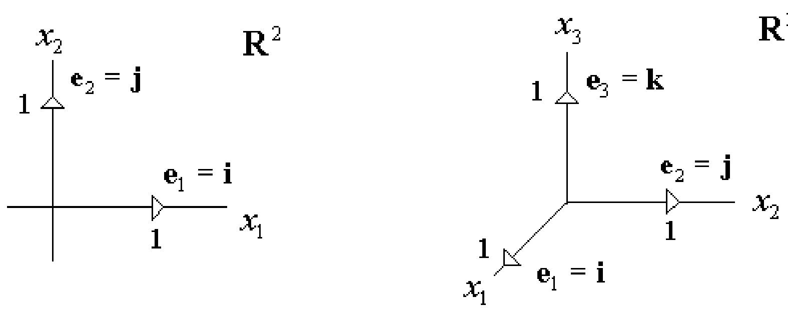



**Fig.:** Standard basis vectors in $\mathbf{R}^2$ and in $\mathbf{R}^3$.

Every $3$-vector can be written in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

## 5.1 Dot Vector Product

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \\ &= u_1 v_1 + \cdots + u_n v_n \end{aligned}$$

**Example 23**

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = (-3)\,4 + 2\,(-1) + (1)\,(5) = -9$$

The **norm**, or **length**, or **magnitude** of an $n$-vector $\mathbf{u}$ is the positive square root

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \left(u_1^2 + \cdots + u_n^2\right)^{\frac{1}{2}}$$

The **(Euclidean) distance** between two $n$-vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} - \mathbf{v}\|$$

A $n$-vector is a **unit** vector, if its norm is $1$.

**Example 24** *Let*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

(a) Find the length of $\mathbf{v}$.

(b) Find the distance between $\mathbf{v}$ and $\mathbf{u}$.

(c) Is $\mathbf{u}$ a unit vector?

Solution: We have

(a) $\|\mathbf{v}\| = \left(1^2 + 2^2 + (-3)^2 + 1^2\right)^{\frac{1}{2}} = \sqrt{15}$

(b) $\|\mathbf{v} - \mathbf{u}\| = \left\|\left(\frac{1}{2}, \frac{5}{2}, -\frac{7}{2}, \frac{3}{2}\right)\right\| = \sqrt{21}$

(c) $\|\mathbf{u}\| = \left\|\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right\| = 1$. So, $\mathbf{u}$ is a unit vector.

**Basic Property**

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \; \|\mathbf{v}\| \cos\theta \tag{2}$$

**Example 25** *Orthogonal vectors.*

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -8 \end{bmatrix} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} = 90^\circ$$

## 5.1 Orthogonal Projections

Let $\mathbf{u}$ and $\mathbf{v}$ be given nonzero vectors. We want to write $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{u}_{\mathrm{pr}} + \mathbf{u}_{\mathrm{c}}$$

where $\mathbf{u}_{\mathrm{pr}}$ is a scalar multiple of $\mathbf{v}$ and $\mathbf{u}_{\mathrm{c}}$ is orthogonal to $\mathbf{u}_{\mathrm{pr}}$ (Fig. 4). This is always possible and such decomposition is unique.

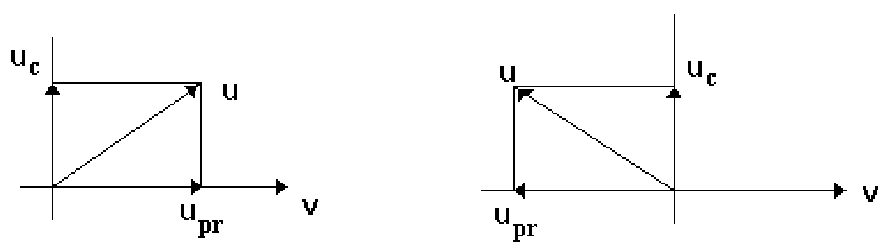



**Fig. 4:** The orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$.

We have

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} \qquad \text{orthogonal projection of } \mathbf{u} \text{ on } \mathbf{v} \tag{3}$$

and

$$\mathbf{u}_{\mathrm{c}} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} \qquad \text{vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{v} \tag{4}$$

**Example 26** *Let* $\mathbf{u} = (1,1,1)$ *and* $\mathbf{v} = (2,2,0)$. *Find the orthogonal projection* $\mathbf{u}_{\text{pr}}$ *of* $\mathbf{u}$ *on* $\mathbf{v}$ *and the vector component* $\mathbf{u}_{\text{c}}$ *of* $\mathbf{u}$ *orthogonal to* $\mathbf{v}$.

Solution: We have

$$\begin{aligned}\mathbf{u}_{\text{pr}} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} = \frac{(1,1,1)\cdot(2,2,0)}{(2,2,0)\cdot(2,2,0)}(2,2,0)\\ &= \frac{4}{8}(2,2,0) = (1,1,0)\end{aligned}$$

and

$$\mathbf{u}_{\text{c}} = \mathbf{u} - \mathbf{u}_{\text{pr}} = (1,1,1) - (1,1,0) = (0,0,1)$$

The answer is geometrically obvious as we see in Fig. Ex. 26.

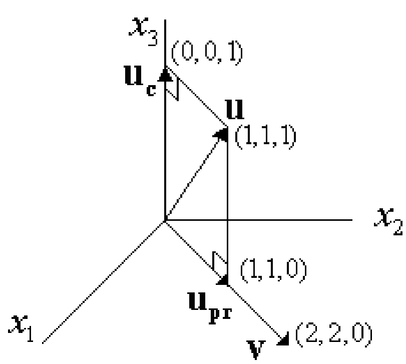



**Fig. Ex. 26:** Projecting $(1,1,1)$ on $(2,2,0)$.

## 5.2 Cross Vector Product

The **cross product** $\mathbf{u} \times \mathbf{v}$ is the *vector* with components

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

This may also be expressed in determinant notation

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

**Example 27** *Find the cross product*

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

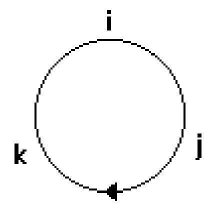
Solution:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\ &= 7\,\mathbf{i} + 5\,\mathbf{j} - 3\,\mathbf{k} \\ &= \begin{bmatrix} 7 \\ 5 \\ -3 \end{bmatrix} \end{aligned}$$

Note that

$$\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

As we move clockwise the cross product of two vectors gives the third. As we move counterclockwise the cross product of two vectors gives the opposite of the third.



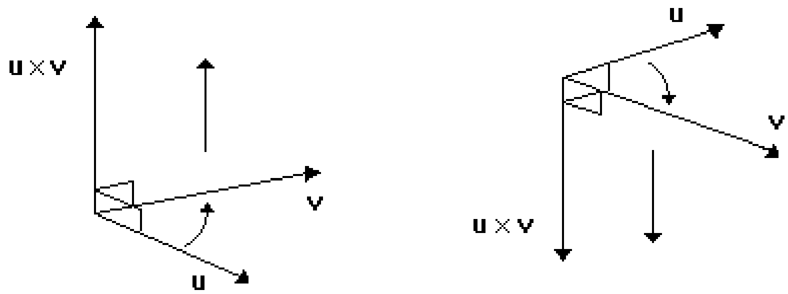

Note that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \qquad \text{and} \qquad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$$

So, $\mathbf{u} \times \mathbf{v}$ is *orthogonal* to $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \perp (\mathbf{u} \times \mathbf{v}) \qquad \text{and} \qquad \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors then the direction of $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane defined by $\mathbf{u}$ and $\mathbf{v}$. Furthermore, it can be shown that for a right-handed coordinate system the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ form also a right–handed system. This determines the direction of the cross product. Next, we determine its length.



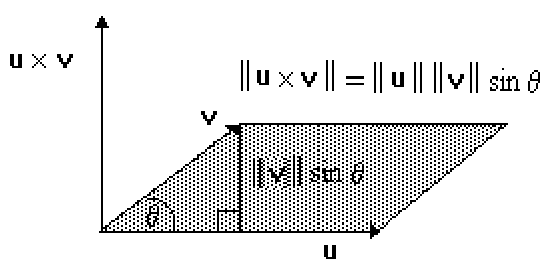

Direction of the cross product for a right–handed system.

**Basic Property**

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \ \|\mathbf{v}\| \sin \theta$$

Geometrically, this length is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$. Hence the area, $A$, of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is

$$A = \|\mathbf{u} \times \mathbf{v}\|$$



## 5.3 Applications of the Cross Product to Geometry

**Example 28 (Area of Parallelogram)** *Compute the area of the parallelogram with adjacent sides $PQ$ and $PR$, where $P(2, 1, 0)$, $Q(1, -2, 1)$ and $R(-2, 2, 4)$.*

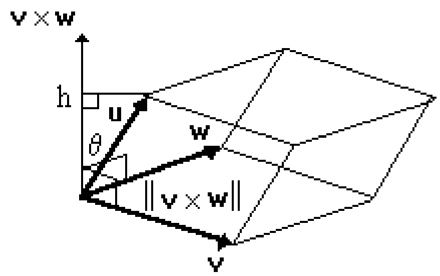
Solution:

$$\begin{aligned} \left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| &= \|(-1, -3, 1) \times (-4, 1, 4)\| \\ &= \|(-13, 0, -13)\| = 13\sqrt{2} \end{aligned}$$

**Example 29 (Area of Triangle)** *Compute the area of the triangle with vertices the tips of* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$.

Solution: $\mathbf{j} - \mathbf{i}$ and $\mathbf{k} - \mathbf{i}$ are two sides of the triangle. Therefore, $\|(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i})\|$ is the area of the parallelogram defined by these sides. One–half of that is the area of the triangle.

$$\begin{aligned} \frac{1}{2} \|(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i})\| &= \frac{1}{2} \|(-1, 1, 0) \times (-1, 0, 1)\| \\ &= \frac{1}{2} \|(1, 1, 1)\| = \frac{1}{2}\sqrt{3} \end{aligned}$$




**Theorem 2 (Volume of Parallelepiped)** *The volume $V$ of the parallelepiped with adjacent sides the position vectors* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *is given by*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \pm \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \tag{5}$$

**Example 30** *Compute the volume of the parallelepiped with adjacent sides the position vectors* $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (0, 2, 1)$ *and* $\mathbf{w} = (3, -2, -1)$.

Solution: We have,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & -2 & -1 \end{vmatrix} = -15$$

Hence, the volume $V$ of the parallelepiped is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-15| = 15$.

# 6 Series

## 6.1 Operations

### Addition

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$

### Scalar Multiplication

$$c \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (ca_n)$$

### Series Multiplication

$$\textstyle\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n$$

$$c_n = \textstyle\sum_{i=1}^{n-1} a_i b_{n-i}$$

$$c_n = a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1$$

## 6.2 The Geometric Series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \textstyle\sum_{n=1}^{\infty} ar^{n-1}$$

$$= \textstyle\sum_{n=0}^{\infty} r^n$$

$$= a \textstyle\sum_{n=0}^{\infty} r^n$$

A geometric series converges to the number $\dfrac{a}{1-r}$, if and only if $|r| < 1$.

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}, \qquad |r| < 1$$

The geometric series is detected by checking in

$$\sum_{n=1}^{\infty} a_n$$

the equality of the ratios

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots$$

If these relations hold then the series is geometric with

$$a = a_1, \qquad r = \frac{a_2}{a_1}$$

**Example 31** *Compute the infinite sum.*

$$3 - \frac{15}{8} + \frac{75}{64} - \frac{375}{512} + \cdots$$

Solution: Because

$$\frac{-\frac{15}{8}}{3} = \frac{\frac{75}{64}}{-\frac{15}{8}} = \frac{-\frac{375}{512}}{\frac{75}{64}} = \cdots = -\frac{5}{8}$$

The series is geometric with $r = -\frac{5}{8}$ and $a = 3$. So it converges to

$$\frac{a}{1-r} = \frac{3}{1-\left(-\frac{5}{8}\right)} = \frac{24}{13}$$

**Example 32** *Write the following number as a rational number (quotient of two integers).*

$$1.222222\ldots$$

Solution:

$$\begin{aligned} 1.222222\ldots &= 1 + \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \frac{2}{10^4} + \cdots \\ &= 1 + \frac{\frac{2}{10}}{1 - \frac{1}{10}} \\ &= \frac{11}{9} \end{aligned}$$

## 6.3 Taylor Series

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

If $a = 0$ we have **Maclaurin Series**

$$
\begin{aligned}
f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n
\end{aligned}
$$

**Example 33** *Find the Maclaurin series for* $\sin(x)$.

Solution:

$$
\begin{array}{rcrrcr}
f(x) & = & \sin(x) & f(0) & = & 0 \\
f'(x) & = & \cos(x) & f'(0) & = & 1 \\
f''(x) & = & -\sin(x) & f''(0) & = & 0 \\
f'''(x) & = & -\cos(x) & f'''(0) & = & -1 \\
f^{(4)}(x) & = & \sin(x) & f^{(4)}(0) & = & 0 \\
 & \vdots & & & \vdots &
\end{array}
$$

The coefficients are

$$0,\ 1,\ 0,\ -1, 0,\ 1,\ 0,\ -1, 0,\ 1,\ 0,\ -1 \ldots$$

So we have

$$
\begin{aligned}
\sin(x) &= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}
\end{aligned}
$$

# 7 Numerical Methods

## 7.1 Bisection Method

**Finding roots of functions iteratively:**

Start with a function $f(x)$ and two points $L_0$ and $R_0$ such that $f(L_0)$ and $f(R_0)$ have opposite signs.

For $n = 0, 1, 2, \ldots$, perform the following steps until sufficient accuracy is achieved.

(1) Set $m = \frac{1}{2}(L_n + R_n)$

(2) find $f(m)$

(3) If $f(L_n) f(m) \leq 0$, set $L_{n+1} = L_n$ and $R_{n+1} = m$. Otherwise set $L_{n+1} = m$ and $R_{n+1} = R_n$.

(4) $f(x)$ has at least one root in the interval $(L_{n+1}, R_{n+1})$. The estimated value of the root is $x^*$

$$x^* = \frac{1}{2}(L_{n+1} + R_{n+1})$$

The maximum error is

$$\frac{1}{2}(R_{n+1} - L_{n+1})$$

**Example 34** *Use two iterations of the Bisection method to find a root of*

$$f(x) = x^3 - 2x - 7$$

Solution: The first step is to find $L_0$ and $R_0$ such that $f(L_0)$, $f(R_0)$ have opposite signs. A table of values of $f(x)$ for random values of $x$ is

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-11$ | $-6$ | $-7$ | $-8$ | $-3$ | $14$ |

$f(x)$ changes sign between $x = 2$ and $x = 3$. $L_0 = 2$ and $R_0 = 3$.

First iteration, $n = 0$

$$m = \frac{1}{2}(2+3) = 2.5$$

$$f(2.5) = (2.5)^3 - 2(2.5) - 7 = 3.625$$

Since $f(2.5) > 0$, a root must exist in $(2, 2.5)$. At this point the best estimate of the root is

$$x^* = \frac{1}{2}(2+2.5) = 2.25$$

and maximum error

$$\frac{1}{2}(2.5-2) = 0.25$$

Second iteration, $n = 1$

$$m = \frac{1}{2}(2+2.5) = 2.25$$

$$f(2.5) = (2.25)^3 - 2(2.25) - 7 = -0.1094$$

Since $f(2.25) < 0$, a root must exist in $(2.25, 2.5)$. At this point the best estimate of the root is

$$x^* = \frac{1}{2}(2.5 + 2.25) = 2.2.375$$

and maximum error

$$\frac{1}{2}(2.5 - 2.25) = 0.125$$

## 7.2 Newton's Method

**Finding roots of functions iteratively:**

Start with a function $f(x)$ and a fist guess $x_0$ for a root. Then for $n = 0, 1, 2, \ldots$ perform the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Example 35** *Use two iterations of Newton's method to find a root of*

$$f(x) = x^3 - 2x - 7$$

Solution: We start with $x_0 = 2$.

$$f'(x) = 3x^2 - 2$$

First iteration, $n = 0$

$$\begin{aligned}
x_0 &= 2 \\
f(2) &= (2)^3 - 2(2) - 7 = -3 \\
f'(2) &= 3(2)^2 - 2 = 10 \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-3}{10} = 2.3
\end{aligned}$$

Second iteration, $n = 1$

$$\begin{aligned}
x_1 &= 2.3 \\
f(2) &= (2.3)^3 - 2(2.3) - 7 = 0.567 \\
f'(2) &= 3(2.3)^2 - 2 = 13.87 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.3 - \frac{0.567}{13.87} = 2.259
\end{aligned}$$

## 7.3 Euler's Method

Euler's method is used to approximate solution to first order initial value problem (IVP).

$$\frac{dx}{dt} = f(t, x), \qquad y(t_0) = x_0$$

The iterations for step size $\Delta t$ are

$$\begin{aligned} t_{n+1} &= t_n + \Delta t \\ x_{n+1} &= x_n + \Delta t f(t_n, x_n) \end{aligned}$$

**Example 36** *Approximate* $x(1.5)$ *with step size* $\Delta t = 0.25$ *for the IVP*

$$\frac{dx}{dt} = 2x, \qquad x(1) = 1$$

Solution: We have

$$x_{n+1} = x_n + \Delta t * f(t_n, x_n)$$

$$x_1 = x_0 + \Delta t (2x_0) = 1 + (0.25)(2(1)) = 1.5$$

$$t_1 = t_0 + \Delta t = 1 + 0.25 = 1.25$$

and iterate again

$$x_2 = x_1 + \Delta t (2x_1) = 1.5 + (0.25)(2(1.5)) = 2.25$$

So in two steps

$$x(1.5) \simeq 2.25$$